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A New Approach to Multiobjective Programming with a Modified Objective Function

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Abstract. In this paper, optimality for multiobjective programming problems having invex objective and constraint functions (with respect to the same function η) is considered. An equivalent vector programming problem is constructed by a modification of the objective function. Furthermore, an η-Lagrange function is introduced for a constructed multiobjective problem and modified saddle point results are presented.

Key words: Efficient point, Invex function with respect to η, Modified vector valued saddle point, Multiobjective programming, Vector valued η-Lagrange function

1. Introduction

An important concept in mathematical models in economics, decision theory, optimal control, and game theory is that of a vector minimum or Pareto optimum. The optimality conditions of Karush–Kuhn–Tucker type for a multiobjective programming problem and the saddle points of its vector-valued Lagrangian function have been studied by many authors (see, for example, [4, 8, 10, 13, 15, 17, 18, 21, 23], and others). But in most of the studies, an assumption of convexity on the functions involving was made. Recently, several new concepts concerning a generalized convex function have been proposed. Among these, the concept of invexity has received more attention [12]. A few authors intended the relevant results in the theory of multiobjective optimization with this concept. For example, Egudo and Hanson [9] have studied a multiobjective programming problem with Mond–Weir type and Wolfe type duals for invex objective and quasi-invex constraint functions. Weir [22] considered a multiobjective programming problem involving invex functions and obtained Karush–Kuhn–Tucker type necessary and sufficient conditions for a feasible point to be properly efficient.

The aim of the present paper is to show how one can obtain optimality conditions for Pareto optimality by constructing for a considered multiobjective programming problem an equivalent vector minimization problem and then using an invexity concept in mathematical programming. The equivalent vector valued problem is obtained by a modification the various objective functions in the given multiobjective programming problem at an arbitrary but fixed point $\overline{x}$. This construction depends heavily on results proved in this paper which connects the efficient points of the original vector minimization problem to the efficient points of the modified multiobjective programming problem. In this way, we obtain a multiobjective programming problem with the same optimality solutions and the value optimality equal to zero. Furthermore, we introduce a definition of an η-Lagrange function in such vector optimization problem, for which modified vector valued saddle points results are presented.

2. Preliminaries

For any $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, ..., n$;
(iii) $x \leqq y$ if and only if $x_i \leqslant y_i$ for all $i = 1, 2, ..., n$;
(iv) $x \leqslant y$ if and only if $x \leqslant y$ and $x \neq y$.

Throughout the paper, we will use the same notation for row and column vectors when the interpretation is obvious.

We consider the multiobjective programming problem

$$\begin{array}{ll} V\text{-minimize} & f(x) = (f_1(x), ..., f_k(x)) \\ \text{subject to} & g_j(x) \leqq 0, \quad j = 1, ..., m, \end{array} \tag{VP}$$

where $f : X \to R^k$ and $g : X \to R^m$ are differentiable functions on a nonempty open set $X \subset R^n$. Note here that the symbol "V-minimize" stands for vector minimization. Let

$$D := \left\{x \in X : g_j(x) \leqq 0, \quad j = 1, ..., m\right\}$$

denote the set of all feasible solutions of (VP).

We define a Lagrange function for the original multiobjective problem (VP)

$$L(x, \lambda, \xi) := \lambda^T f(x) + \xi^T g(x).$$

For such optimization problems minimization means obtaining of efficient solutions (Pareto optimal solutions) in the following sense [17]:

DEFINITION 1. A point $\overline{x} \in D$ is said to be an efficient (Pareto optimal) point for (VP) if and only if there exists no $x \in D$ such that

$$f(x) \leqslant f(\overline{x}).$$

It is said to be a weak efficient (weak Pareto optimal) point for (VP) if and only if there exists no $x \in D$ such that

$$f(x) < f(\overline{x}).$$

Speaking roughly, a point $\overline{x} \in D$ is efficient (Pareto optimal) for a multiobjective problem (VP) if and only if we can improve (in the sense of minimization) the value of one of the objective functions only at the cost of making at least one of the remaining objective functions worse; it is weak efficient (weak Pareto optimal) if and only if we cannot improve further all of the objective functions simultaneously.

To prove various results in the paper we need certain optimality conditions for the multiobjective problem (VP). Necessary optimality conditions a Karush–Kuhn–Tucker type for the multiobjective problems, by using some regularity conditions, were obtained, for example, by Craven [6], Giorgi and Guerraggio [11], Kanniappan [15], Singh [19], Weir et al. [21]. Note that we are dealing with the multiobjective problems in finite dimensional spaces only assuming that the involved functions are differentiable. Therefore, we are using the following necessary optimality conditions of Karush–Kuhn–Tucker type for a such multiobjective programming under some constraint qualification (CQ) (for example, Linear Independence Constraint Qualification [2]):

THEOREM 2. [19] *Let $\overline{x}$ be an (weak) efficient point in (VP) and some constraint qualification (CQ) holds for (VP). Then there exist $\overline{\lambda} \in R^k_+$, $\overline{\lambda} \neq 0$ and $\overline{\xi} \in R^k_+$, $\overline{\xi} \geqq 0$, such that*

$$\overline{\lambda}^T \nabla f\,(\overline{x}) + \overline{\xi}^T \nabla g\,(\overline{x}) = 0, \tag{1}$$

$$\overline{\xi}^T g\,(\overline{x}) = 0. \tag{2}$$

To make things easier, we consider the invexity and generalized invexity definitions for vectorial functions, which coincide with those given in the scalar case (see [12]).

DEFINITION 3. Let $f : X \to R^k$ be a differentiable function on a nonempty open set $X \subset R^n$. Then, f is invex with respect to η at $u \in X$ on X if, for all $x \in X$, there exists $\eta : X \times X \to R^n$ such that

$$f(x) - f(u) \geqq \nabla f(u)\eta\,(x, u)\,. \tag{3}$$

If the inequality (3) holds for any $u \in X$ then f is invex with respect to η on X.

DEFINITION 4. Let $f : X \to R^k$ be a differentiable function on a nonempty open set $X \subset R^n$. Then, f is strictly invex with respect to η at $u \in X$ on X if, for all $x \in X$ with $x \neq u$, there exists $\eta : X \times X \to R^n$ such that

$$f(x) - f(u) > \nabla f(u)\eta\,(x, u) \tag{4}$$

If the inequality (4) holds for any $u \in X$ then f is strictly invex with respect to η on X.

DEFINITION 5. Let $f : X \to R^k$ be a differentiable function on a nonempty open set $X \subset R^n$. Then, f is pseudo-invex with respect to η at $u \in X$ on X if, for all $x \in X$, there exists $\eta : X \times X \to R^n$ such that

$$f(x) - f(u) < 0 \Longrightarrow \nabla f(u)\eta(x, u) < 0. \tag{5}$$

If the inequality (5) holds for any $u \in X$ then f is pseudo-invex with respect to η on X.

DEFINITION 6. Let $f : X \to R^k$ be a differentiable function on a nonempty open set $X \subset R^n$. Then, f is strictly pseudo-invex with respect to η at $u \in X$ on X if, for all $x \in X$, $x \neq u$, there exists $\eta : X \times X \to R^n$ such that

$$f(x) - f(u) \leqq 0 \Longrightarrow \nabla f(u)\eta(x, u) < 0. \tag{6}$$

DEFINITION 7. Let $f : X \to R^k$ be a differentiable function on a nonempty open set $X \subset R^n$. Then, f is quasi-invex with respect to η at $u \in X$ on X if, for all $x \in X$, there exists $\eta : X \times X \to R^n$ such that

$$f(x) - f(u) \leqq 0 \Longrightarrow \nabla f(u)\eta(x, u) \leqq 0. \tag{7}$$

If the inequality (7) holds for any $u \in X$ then f is quasi-invex with respect to η on X.

It is clear that

$$\text{invexity} \Longrightarrow \text{pseudo-invexity} \Longrightarrow \text{quasi-invexity}. \tag{8}$$

3. An equivalent multiobjective problem and optimality conditions

Let $\overline{x}$ be a feasible solution in (VP). We consider the following multiobjective program (VP$_\eta(\overline{x})$) given by

$$\begin{array}{ll} V\text{-minimize} & \left(\left[\eta(x,\overline{x})\right]^T \nabla f_1(\overline{x}), \ldots, \left[\eta(x,\overline{x})\right]^T \nabla f_k(\overline{x})\right) \\ \text{subject to} & g_j(x) \leqq 0, \quad j = 1, \ldots, m, \end{array} \tag{VP$_\eta(\overline{x})$}$$

where f, g, X are defined as in (VP) and η is a vector-valued function defined as $\eta : D \times D \to R^n$.

THEOREM 8. *Let $\overline{x}$ be (weak) efficient in (VP) and (CQ) holds at $\overline{x}$ for (VP). Further, we assume that g is invex with respect to η at $\overline{x}$ on D and $\eta(\overline{x}, \overline{x}) = 0$. Then $\overline{x}$ is (weak) efficient in VP$_\eta(\overline{x})$.*

Proof. Since $\overline{x}$ is efficient in (VP) and (CQ) holds at $\overline{x}$ for (VP), then Karush–Kuhn–Tucker conditions (1)–(2) are satisfied. We proceed by contradiction. Let $\overline{x}$ not be efficient for (VP$_\eta(\overline{x})$). This implies that there exists $\widetilde{x}$ feasible for (VP$_\eta(\overline{x})$) (and so to (VP)) such that

$$\left[\eta(\widetilde{x}, \overline{x})\right]^T \nabla f_i(\overline{x}) \leqq \left[\eta(\overline{x}, \overline{x})\right]^T \nabla f_i(\overline{x}) \text{ for all } i = 1, \ldots, k, \tag{9}$$

$$\left[\eta(\widetilde{x}, \overline{x})\right]^T \nabla f_s(\overline{x}) < \left[\eta(\overline{x}, \overline{x})\right]^T \nabla f_s(\overline{x}) \text{ for some } s \in \{1, \ldots, k\}. \tag{10}$$

From (9) and (10) together with assumption $\eta(\overline{x}, \overline{x}) = 0$, we get

$$\left[\eta(\widetilde{x}, \overline{x})\right]^T \nabla f_i(\overline{x}) \leqq 0 \text{ for all } i = 1, \ldots, k, \tag{11}$$

$$\left[\eta(\widetilde{x}, \overline{x})\right]^T \nabla f_s(\overline{x}) < 0 \text{ for some } s \in \{1, \ldots, k\}. \tag{12}$$

Since $\lambda > 0$, by (11), (12) we obtain

$$\lambda^T \nabla f(\overline{x}) \eta(\widetilde{x}, \overline{x}) < 0. \tag{13}$$

A feasibility of $\widetilde{x}$ together with $\xi \geqq 0$ implies that $\xi^T g(\widetilde{x}) \leqq 0$. Hence by (2), it follows that $\xi^T g(\widetilde{x}) \leqq \xi^T g(\overline{x})$. By assumption, g is invex with respect to η at $\overline{x}$ on D. Thus

$$\xi^T \nabla g(\overline{x}) \eta(\widetilde{x}, \overline{x}) \leqq 0. \tag{14}$$

Adding (13) and (14), we obtain the inequality

$$\left[\lambda^T \nabla f(\overline{x}) + \xi^T \nabla g(\overline{x})\right] \eta(\widetilde{x}, \overline{x}) < 0,$$

which contradicts (1). Hence $\overline{x}$ is efficient in (VP$_\eta(\overline{x})$).
Proof for weak efficiency is similar. □

THEOREM 9. *Let $\overline{x}$ be a feasible point for (VP$_\eta(\overline{x})$). Further, we assume that f is invex with respect to η at $\overline{x}$ on D and $\eta(\overline{x}, \overline{x}) = 0$. If $\overline{x}$ is efficient in (VP$_\eta(\overline{x})$) then $\overline{x}$ is also efficient in (VP).*

Proof. We proceed by contradiction. Let $\overline{x}$ be no efficient in (VP). Then there exists $\widetilde{x}$ feasible for (VP) such that

$$f_i(\widetilde{x}) \leqq f_i(\overline{x}) \text{ for all } i = 1, \ldots, k, \tag{15}$$

$$f_s(\widetilde{x}) < f_s(\overline{x}) \text{ for some } s \in \{1, \ldots, k\}. \tag{16}$$

By assumption f_i, $i = 1, ..., k$, are invex with respect to η at $\overline{x}$ on D. It follows that they are also quasi-invex with respect to η at $\overline{x}$ on D. Therefore, (15) gives

$$\left[\eta\left(\widetilde{x}, \overline{x}\right)\right]^T \nabla f_i(\overline{x}) \leqq 0 \text{ for all } i = 1, ..., k. \tag{17}$$

Since f_i, $i = 1, ..., k$, are invex with respect to η at $\overline{x}$ on D, it follows that they are also pseudo-invex with respect to η at $\overline{x}$ on D. Therefore, (16) gives

$$\left[\eta\left(\widetilde{x}, \overline{x}\right)\right]^T \nabla f_s(\overline{x}) < 0 \text{ for some } s \in \{1, ..., k\}. \tag{18}$$

By assumption, $\eta\left(\overline{x}, \overline{x}\right) = 0$. Hence from (17) together with (18) we obtain

$$\left[\eta\left(\widetilde{x}, \overline{x}\right)\right]^T \nabla f_i(\overline{x}) \leqq \left[\eta\left(\overline{x}, \overline{x}\right)\right]^T \nabla f_i(\overline{x}) \text{ for all } i = 1, ..., k, \tag{19}$$

$$\left[\eta\left(\widetilde{x}, \overline{x}\right)\right]^T \nabla f_s(\overline{x}) < \left[\eta\left(\overline{x}, \overline{x}\right)\right]^T \nabla f_i(\overline{x}) \text{ for some } s \in \{1, ..., k\}, \tag{20}$$

which contradicts that $\overline{x}$ is efficient in (VP$_\eta(\overline{x})$). Hence the theorem is proved. □

In view of Theorem 8 and Theorem 9, if we assume that f and g are invex with respect to the same function η at $\overline{x}$ on the set of feasible solutions D and $\eta\left(\overline{x}, \overline{x}\right) = 0$ then multiobjective programming problems (VP) and (VP$_\eta(\overline{x})$) are equivalent in the sense discussed above.

Now, we prove this theorem under the weakened assumption on the functions involving. This follows from the proof of Theorem 9, in which, in fact, we used the assumption of generalized invexity (that is, pseudo-invexity and quasi-invexity). Therefore, we replace the invexity assumption of f by (pseudo-invexity) strict pseudo-invexity to prove the relationship between (weak) efficient points of the modified multiobjective problem (VP$_\eta(\overline{x})$) and the original multiobjective problem (VP).

THEOREM 10. *Let $\overline{x}$ be a feasible point for (VP$_\eta(\overline{x})$). Further, we assume that f is (pseudo-invex) strictly pseudo-invex with respect to η at $\overline{x}$ on D and $\eta\left(\overline{x}, \overline{x}\right) = 0$. If $\overline{x}$ is a (weak) efficient point in (VP$_\eta(\overline{x})$) then $\overline{x}$ is also a (weak) efficient point in (VP).*

REMARK 11. If a function $\eta : D \times D \to R^n$ (with respect to which f and g are invex) is linear with respect to the first component and, moreover, g is a linear function, then (VP$_\eta(\overline{x})$) is a linear multiobjective programming problem. Now we give an example of a multiobjective programming problem which by using the approach discussed in this paper is transformed to a linear multiobjective programming problem (VP$_\eta(\overline{x})$).

EXAMPLE 12. We consider the following multiobjective programming problem

$$\begin{aligned} f(x) = \left(\tfrac{1}{3}x_1^3 - \tfrac{1}{2}x_1^2 + 5x_1 + \tfrac{1}{6}, 5x_1 + e^{x_2}\right) &\to \min \\ g_1(x) = 1 - x_1 &\leqslant 0, \\ g_2(x) = 1 - x_2 &\leqslant 0. \end{aligned}$$

Note that $\overline{x} = (1, 1)$ is an efficient point in the considered problem. Further, it can be proved that f and g are invex at $\overline{x}$ with respect to the same function η defined by

$$\eta(x, u) = \begin{bmatrix} \frac{x_1 - u_1}{5} \\ \frac{x_2 - u_2}{e^{u_2}} \end{bmatrix}.$$

Now using the approach discussed in the paper we construct the problem (VP$_\eta(\overline{x})$) by transforming the objective function. Thus, we obtain a linear multiobjective programming problem in the form

$$\begin{aligned} &(x_1 - 1, x_1 + x_2 - 2) \to \min \\ &\quad g_1(x) = 1 - x_1 \leqslant 0, \\ &\quad g_2(x) = 1 - x_2 \leqslant 0. \end{aligned}$$

It is not difficult to see, that $\overline{x} = (1, 1)$ is also efficient in the above optimization problem, that is, in the multiobjective optimization problem which is constructed by a modification of the objective function in the original problem.

REMARK 13. The assumption that a function η satisfies the condition $\eta(\overline{x}, \overline{x}) = 0$ is essential to confirm the equivalency between the multiobjective programming problems (VP) and (VP$_\eta(\overline{x})$) in the sense discussed in the paper. In the example below we show that in the case when this condition does not hold then we have no equivalency between (VP) and (VP$_\eta(\overline{x})$).

EXAMPLE 14. We consider the following multiobjective programming problem

$$\begin{aligned} &f(x) = \left(\ln(x_1), \sqrt{x_2}\right) \to \min \\ &\quad g_1(x) = 1 - x_1 \leqslant 0, \\ &\quad g_2(x) = 1 - x_2 \leqslant 0. \end{aligned}$$

Note that $\overline{x} = (1, 1)$ is an efficient point in the considered problem. Further, it can be proved that f and g are invex at $\overline{x}$ with respect to the same function η defined by

$$\eta(x, u) = \begin{bmatrix} \frac{1}{2}u_1 - x_1 \\ -x_2 - u_2 \end{bmatrix}.$$

For the considered multiobjective programming problem we construct the transformed multiobjective programming problem (VP$_\eta(\overline{x})$). We have

$$\begin{aligned} &\left(\tfrac{1}{2} - x_1, -x_2 - 1\right) \to \min \\ &\quad g_1(x) = 1 - x_1 \leqslant 0, \\ &\quad g_2(x) = 1 - x_2 \leqslant 0. \end{aligned}$$

But this multiobjective objective problem is unbounded on the set of feasible solutions. Thus, the considered multiobjective programming problems are no equivalent in the sense discussed in the paper.

4. Saddle criteria

Now we introduce a definition of an η-Lagrange function for a multiobjective programming problem (VP$_\eta(\overline{x})$).

DEFINITION 15. *An η-Lagrange function is said to be a Lagrange function for a multiobjective programming problem (VP$_\eta(\overline{x})$)*

$$L_\eta(x,\xi) \ := \left[\eta(x,\overline{x})\right]^T \nabla f(\overline{x}) + \xi^T g(x)$$
$$:= \left(\left[\eta(x,\overline{x})\right]^T \nabla f_1(\overline{x}) + \xi^T g(x), ..., \left[\eta(x,\overline{x})\right]^T \nabla f_k(\overline{x}) + \xi^T g(x)\right).$$

For a Lagrange function, some kinds of saddle points have been introduced, such as those in [20]. Here, we give a new definition of a (Pareto) saddle point for the introduced η-Lagrange function in a multiobjective programming problem (VP$_\eta(\overline{x})$).

DEFINITION 16. A point $\left(\overline{x},\overline{\xi}\right) \in D \times R^m_+$ is said to be a (Pareto) saddle point for the η-Lagrange function if

(i) $L_\eta(\overline{x},\xi) \leqq L_\eta\left(\overline{x},\overline{\xi}\right), \quad \forall \xi \in R^m_+$,
(ii) $L_\eta\left(x,\overline{\xi}\right) \nless L_\eta\left(\overline{x},\overline{\xi}\right), \quad \forall x \in D$.

THEOREM 17. *We assume that f is (invex) strictly invex with respect to η at $\overline{x}$ on D with $\eta(\overline{x},\overline{x}) = 0$ and some constraint qualification (CQ) holds at $\overline{x}$ for (VP). If $\left(\overline{x},\overline{\xi}\right)$ is a saddle point for L_η, then $\overline{x}$ is a (weak) Pareto solution in (VP).*

Proof. We assume that $\left(\overline{x},\overline{\xi}\right)$ is a saddle point for L_η. Then by i) we have

$$\left[\eta(\overline{x},\overline{x})\right]^T \nabla f(\overline{x}) + \xi^T g(\overline{x}) \leqq \left[\eta(\overline{x},\overline{x})\right]^T \nabla f(\overline{x}) + \overline{\xi}^T g(\overline{x}), \quad \forall \xi \in R^m_+.$$

Since $\eta(\overline{x},\overline{x}) = 0$, therefore

$$\xi^T g(\overline{x}) \leqq \overline{\xi}^T g(\overline{x}), \ \forall \xi \in R^m_+. \tag{21}$$

We proceed by contradiction, that is, suppose that $\overline{x}$ is not a weak Pareto solution in (VP). Then there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) < f(\overline{x}). \tag{22}$$

Since $\overline{x} \in D$ and $\overline{\xi} \in R^m_+$, then we have

$$\overline{\xi}^T g(\overline{x}) \leqq 0. \tag{23}$$

In (21), let $\xi = 0$

$$\overline{\xi}^T g(\overline{x}) \geqq 0. \tag{24}$$

Hence, (23) together with (24) gives

$$\overline{\xi}^T g(\overline{x}) = 0. \tag{25}$$

Since f is invex with respect to η on D, then it is also pseudo-invex with respect to the same function η on D. Hence by (22), it follows that

$$\left[\eta\left(\widetilde{x}, \overline{x}\right)\right]^T \nabla f(\overline{x}) < 0. \tag{26}$$

Thus, by (25) and (26) and using the definition of L_η, we get

$$\begin{aligned} L_\eta\left(\widetilde{x}, \overline{\xi}\right) &= \left[\eta\left(\widetilde{x}, \overline{x}\right)\right]^T \nabla f(\overline{x}) + \overline{\xi}^T g(\widetilde{x}) < \left[\eta\left(\overline{x}, \overline{x}\right)\right]^T \nabla f(\overline{x}) + \overline{\xi}^T g(\overline{x}) \\ &= L_\eta\left(\overline{x}, \overline{\xi}\right) \end{aligned}$$

This contradicts ii), therefore, $\overline{x}$ is a weak Pareto solution in (VP).

The proof of efficiency is similar. □

Now we prove a converse theorem, that is, a sufficient condition for a point $\left(\overline{x}, \overline{\xi}\right) \in D \times R^m_+$ to be a saddle point for the η-Lagrange function.

THEOREM 18. *Let $\overline{x}$ be a (weak) Pareto solution in (VP) at which some constraint qualification (CQ) is satisfied. Further, we assume that f and g are invex with respect to the same function η at $\overline{x}$ on D and $\eta\left(\overline{x}, \overline{x}\right) = 0$. Then there exists $\overline{\xi} \in R^m_+$, such that $\left(\overline{x}, \overline{\xi}\right)$ is a saddle point for the η-Lagrange function in a multiobjective programming problem (VP$_\eta(\overline{x})$).*

Proof. By assumption, $\overline{x}$ is a weak Pareto solution for (VP). Thus, by Theorem 2, it follows that Karush–Kuhn–Tucker conditions (1) and (2) hold. Not losing generality of the considerations, we assume $\sum_{i=1}^k \overline{\lambda}_i = 1$. Since g is invex with respect to η at $\overline{x}$ on D and $\overline{\xi} \in R^m_+$, it follows that the inequality

$$\overline{\xi}^T g(x) - \overline{\xi}^T g\left(\overline{x}\right) \geqq \overline{\xi}^T \nabla g\left(\overline{x}\right) \eta\left(x, \overline{x}\right)$$

holds for all $x \in D$. From (1)

$$\overline{\xi}^T g(x) - \overline{\xi}^T g\left(\overline{x}\right) \geqq -\overline{\lambda}^T \nabla f\left(\overline{x}\right) \eta\left(x, \overline{x}\right).$$

By assumption $\eta\left(\overline{x}, \overline{x}\right) = 0$. Thus, the inequality

$$\overline{\lambda}^T \nabla f\left(\overline{x}\right) \eta\left(x, \overline{x}\right) + \overline{\xi}^T g(x) \geqq \overline{\lambda}^T \nabla f\left(\overline{x}\right) \eta\left(\overline{x}, \overline{x}\right) + \overline{\xi}^T g\left(\overline{x}\right)$$

holds for all $x \in D$. Since $\overline{\lambda} \in R^k_+ \backslash \{0\}$, $\sum_{i=1}^k \overline{\lambda}_i = 1$, and by the definition of the η-Lagrange function, it follows that, for all $x \in D$

$$\overline{\lambda}^T L_\eta\left(x, \overline{\xi}\right) \geqq \overline{\lambda}^T L_\eta\left(\overline{x}, \overline{\xi}\right).$$

This means that, the relation

$$\overline{\lambda}^T L_\eta \left(x, \overline{\xi}\right) \nless \overline{\lambda}^T L_\eta \left(\overline{x}, \overline{\xi}\right) \tag{27}$$

holds for all $x \in D$.
By (2) and since $\overline{x} \in D$, the inequality

$$\xi^T g\left(\overline{x}\right) \leqq \overline{\xi}^T g\left(\overline{x}\right)$$

holds for any $\xi \in R^m_+$. Thus the inequality

$$\left[\eta\left(\overline{x}, \overline{x}\right)\right]^T \nabla f\left(\overline{x}\right) + \xi^T g\left(\overline{x}\right) \leqq \left[\eta\left(\overline{x}, \overline{x}\right)\right]^T \nabla f\left(\overline{x}\right) + \overline{\xi}^T g\left(\overline{x}\right).$$

for all $\lambda \in R^k_+$ and $\xi \in R^m_+$. This means, by the definition of η-Lagrange function, that

$$L_\eta\left(\overline{x}, \xi\right) \leqq L_\eta\left(\overline{x}, \overline{\xi}\right). \tag{28}$$

Inequalities (27) and (28) mean that $\left(\overline{x}, \overline{\xi}\right)$ is a saddle point for the η-Lagrange function in a multiobjective programming problem (VP$_\eta(\overline{x})$). □

In view of Theorem 17 and Theorem 18, we see that, if we assume that f is (invex) strictly invex and g is also invex with respect to the same function η at $\overline{x}$ on D, the method of a modified objective function guarantees the equivalency between a (weak) Pareto solution in (VP) and a saddle point of the η-Lagrange function in a modified multiobjective programming problem (VP$_\eta(\overline{x})$) in the sense discussed above.

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